Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 1, February 2003, pp. 57–65.
Printed in India

Fields and forms on ρ-algebras

CĂTĂLIN CIUPALĂ

Department of Differential Equations, Faculty of Mathematics and Informatics,
University Transilvania of Braşov, 2200 Braşov, Romania
E-mail: cciupala@yahoo.com

MS received 13 May 2004; revised 21 October 2004

Abstract. In this paper we introduce non-commutative fields and forms on a new kind of non-commutative algebras: ρ-algebras. We also define the Frölicher–Nijenhuis bracket in the non-commutative geometry on ρ-algebras.

Keywords. Non-commutative geometry; ρ-algebras; Frölicher–Nijenhuis bracket.

1. Introduction

There are some ways to define the Frölicher–Nijenhuis bracket in non-commutative differential geometry. The Frölicher–Nijenhuis bracket on the algebra of universal differential forms of a non-commutative algebra, is presented in [2], the Frölicher–Nijenhuis bracket in several kinds of differential graded algebras are defined in [6] and the Frölicher–Nijenhuis bracket on colour commutative algebras is defined in [7]. But this notion is not defined on ρ-algebras in the context of non-commutative geometry. In this paper we introduce the Frölicher–Nijenhuis bracket on a ρ-algebra A using the algebra of universal differential forms $\Omega^*(A)$.

A ρ-algebra A over the field k ($\mathbb{C}$ or $\mathbb{R}$) is a G-graded algebra (G is a commutative group) together with a twisted cocycle $\rho\colon G\times G\to k$. These algebras were defined for the first time in the paper [1] and are generalizations of usual algebras (the case when G is trivial) and of $\mathbb{Z}$ ($\mathbb{Z}_2$)-superalgebras (the case when G is $\mathbb{Z}$ resp. $\mathbb{Z}_2$). Our construction of the Frölicher–Nijenhuis bracket for ρ-algebras, in this paper, is a generalization of this bracket from [2].

In §2 we present a class of non-commutative algebras which are ρ-algebras, derivations and bimodules. In §3 we define the algebra of (non-commutative) universal differential forms $\Omega^*(A)$ of a ρ-algebra A. In §4 we present the Frölicher–Nijenhuis calculus on A, the Nijenhuis algebra of A, and the Frölicher–Nijenhuis bracket on A. We also show the naturality of the Frölicher–Nijenhuis bracket.

2. ρ-Algebras

In this section we present a class of non-commutative algebras that are ρ-algebras. For more details see [1].

Let G be an abelian group, additively written, and let A be a G-graded algebra. This implies that the vector space A has a G-grading $A=\oplus_{a\in G}A_a$, and that $A_aA_b\subset A_{a+b}$

$(a,b \in G)$. The G-degree of a (non-zero) homogeneous element f of A is denoted as $|f|$. Futhermore let $\rho\colon G \times G \to k$ be a map which satisfies

$$\rho(a,b) = \rho(b,a)^{-1}, \quad a,b \in G, \tag{1}$$

$$\rho(a+b,c) = \rho(a,c)\rho(b,c), \quad a,b,c \in G. \tag{2}$$

This implies $\rho(a,b) \neq 0$, $\rho(0,b) = 0$ and $\rho(c,c) = \pm 1$ for all $a,b,c \in G$, $c \neq 0$. We define for homogeneous elements f and g in A an expression, which is ρ-commutator of f and g as

$$[f,g]_\rho = fg - \rho(|f|\,|g|)gf. \tag{3}$$

This expression as it stands make sense only for homogeneous elements f and g, but can be extended linearly to general elements. A G-graded algebra A with a given cocycle ρ will be called ρ-commutative if $fg = \rho(|f|,|g|)gf$ for all homogeneous elements f and g in A.

Examples.

1) Any usual (commutative) algebra is a ρ-algebra with the trivial group G.
2) Let $G = \mathbb{Z}$ $(\mathbb{Z}_2)$ be the group and the cocycle $\rho(a,b) = (-1)^{ab}$, for any $a,b \in G$. In this case any ρ-(commutative) algebra is a super(commutative) algebra.
3) The N-dimensional quantum hyperplane [1,3,4] S_N^q, is the algebra generated by the unit element and N linearly independent elements $x_1,\dots,x_N$ satisfying the relations:

   $$x_i x_j = q x_j x_i, \quad i < j$$

   for some fixed $q \in k$, $q \neq 0$. S_N^q is a $\mathbb{Z}^N$-graded algebra, i.e.,

   $$S_N^q = \bigoplus_{n_1,\dots,n_N}^{\infty} (S_N^q)_{n_1\dots n_N},$$

   with $(S_N^q)_{n_1\dots n_N}$ the one-dimensional subspace spanned by products $x^{n_1}\cdots x^{n_N}$. The $\mathbb{Z}^N$-degree of these elements is denoted by

   $$|x^{n_1}\cdots x^{n_N}| = n = (n_1,\dots,n_N).$$

   Define the function $\rho\colon \mathbb{Z}^N \times \mathbb{Z}^N \to k$ as

   $$\rho(n,n') = q^{\sum_{j,k=1}^N n_j n'_k \alpha_{jk}},$$

   with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$. It is obvious that S_N^q is a ρ-commutative algebra.
4) The algebra of matrix $M_n(\mathbb{C})$ [5] is ρ-commutative as follows:
   Let

   $$p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^2 & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \varepsilon^{n-1} & 0 \end{pmatrix},$$

   $p,q \in M_n(\mathbb{C})$, where $\varepsilon^n = 1$, $\varepsilon \neq 1$. Then $pq = \varepsilon qp$ and $M_n(\mathbb{C})$ is generated by the set $B = \{p^a q^b | a,b = 0,1,\dots,n-1\}$.

It is easy to see that $p^a q^b = \varepsilon^{ab} q^b p^a$ and $q^b p^a = \varepsilon^{-ab} p^a q^b$ for any $a, b = 0, 1, \ldots, n-1$. Let $G := \mathbb{Z}_n \oplus \mathbb{Z}_n$, $\alpha = (\alpha_1, \alpha_2) \in G$ and $x_\alpha := p^{\alpha_1} q^{\alpha_2} \in M_n(\mathbb{C})$. If we denote $\rho(\alpha, \beta) = \varepsilon^{\alpha_2\beta_1 - \alpha_1\beta_2}$ then $x_\alpha x_\beta = \rho(\alpha, \beta) x_\beta x_\alpha$, for any $\alpha, \beta \in G$, $x_\alpha, x_\beta \in B$.

It is obvious that the map $\rho\colon G \times G \to \mathbb{C}$, $\rho(\alpha, \beta) = \varepsilon^{\alpha_2\beta_i - \alpha_1\beta_2}$ is a cocycle and that $M_n(\mathbb{C})$ is a ρ-commutative algebra.

Let α be an element of the group G. A ρ-derivation X of A, of degree α is a bilinear map $X\colon A \to A$ of G-degree $|X|$ i.e. $X\colon A_* \to A_{*+|X|}$, such that one has for all elements $f \in A_{|f|}$ and $g \in A$,

$$X(fg) = (Xf)g + \rho(\alpha, |f|) f(Xg). \tag{4}$$

Without any difficulties it can be obtained that if algebra A is ρ-commutative, $f \in A_{|f|}$ and X is a ρ-derivation of degree α, then fX is a ρ-derivation of degree $|f| + \alpha$ and the G-degree $|f| + |X|$ i.e.

$$(fX)(gh) = ((fX)g)h + \rho(|f| + \alpha, |g|) g(fX)h$$

and $fX\colon A_* \to A_{*+|f|+|X|}$.

We say that $X\colon A \to A$ is a ρ-derivation if it has degree equal to G-degree $|X|$ i.e. $X\colon A_* \to A_{*+|X|}$ and $X(fg) = (Xf)g + \rho(|X|, |f|) f(Xg)$ for any $f \in A_{|f|}$ and $g \in A$.

It is known [1] that the ρ-commutator of two ρ-derivations is again a ρ-derivation and the linear space of all ρ-derivations is a ρ-Lie algebra, denoted by ρ-Der A.

One verifies immediately that for such an algebra A, ρ-Der A is not only a ρ-Lie algebra but also a left A-module with the action of A on ρ-Der A defined by

$$(fX)g = f(Xg) \quad f, g \in A,\ X \in \rho\text{-Der}A. \tag{5}$$

Let M be a G-graded left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A, \psi \in M$. Then M is also a right A-module with the right action on M defined by

$$\psi f = \rho(|\psi|, |f|) f \psi. \tag{6}$$

In fact M is a bimodule over A, i.e.

$$f(\psi g) = (f\psi)g \quad f, g \in A,\ \psi \in M. \tag{7}$$

Let M and N be two G-graded bimodules over the ρ-algebra A. Let $f\colon M \to N$ be an A-bimodule homomorphism of degree $\alpha \in G$ if $f\colon M_\beta \to N_{\alpha+\beta}$ such that $f(am) = \rho(\alpha, |a|) a f(m)$ and $f(ma) = f(m)a$ for any $a \in A_{|a|}$ and $m \in M$. We denote by $\mathrm{Hom}_\alpha(M, N)$ the space of A-bimodule homomorphisms of degree α and by $\mathrm{Hom}_A^A(M, N) = \oplus_{\alpha\in G} \mathrm{Hom}_\alpha(M, N)$ the space of all A-bimodule homomorphisms.

3. Differential forms on a ρ-algebra

A is a ρ-algebra as in the previous section. We denote by $\Omega^1_\alpha(A)$ the space generated by the elements: adb of G-degree $|a| + |b| = \alpha$ with the usual relations:

$$d(a+b) = d(a) + d(b), \quad d(ab) = d(a)b + ad(b) \quad \text{and} \quad d1 = 0,$$

where 1 is the unit of the algebra A.

If we denote by $\Omega^1(A)=\sum_\alpha \Omega^1_\alpha(A)$ then $\Omega^1(A)$ is an A-bimodule and satisfies the following theorem of universality.

Theorem 1. *For any A-bimodule M and for any derivation $X\colon A\to M$ of degree $|X|$ there is an A-bimodule homomorphism $f\colon \Omega^1(A)\to M$ of degree $|X|$ ($f\in \mathrm{Hom}_{|X|}(\Omega^1(A),M)$) such that $X=f\circ d$. The homomorphism is uniquely determined and the corresponding $X\mapsto f$ establishes an isomorphism between ρ-$\mathrm{Der}_{|X|}(A,M)$ and $\mathrm{Hom}_{|X|}(\Omega^1(A),M)$.*

Proof. We define the map $f\colon \Omega^1(A)\to M$ by $f(adb)=\rho(|X|,|a|)aX(b)$ which transform the usual Leibniz rule for the operator d into the ρ-Leibniz rule for the derivation X. $\square$

Starting from the A-bimodule $\Omega^1(A)$ and the ρ-algebra $\Omega^0(A)=A$ we build up *the algebra of differential forms over A.*

This algebra will be a new $\overline{\rho}$-algebra

$$\Omega^*(A)=\sum_{n\in\mathbb{N},\alpha\in G}\Omega^n_\alpha(A)$$

graded by the group $\overline{G}=\mathbb{Z}\times G$ and generated by elements $a\in A_{|a|}=\Omega^0_{|a|}(A)$ of degree $(0,|a|)$ and their differentials $da\in\Omega^1_{|a|}(A)$ of degree $(1,|a|)$.

We will also require the universal derivation $d\colon A\to\Omega^1(A)$ which can be extended to a $\overline{\rho}$-derivation of the algebra $\Omega^*(A)$ of degree $(1,0)$ in such a way that $d^2=0$ and $\overline{\rho}|_{G\times G}=\rho$. Denote by $\omega\wedge\theta\in\Omega^{n+m}_{\alpha+\beta}(A)$ the product of forms $\omega\in\Omega^n_\alpha(A)$, $\theta\in\Omega^m_\beta(A)$ in the algebra $\Omega^*(A)$. Then

$$\mathrm{d}(\omega\wedge\theta)=\mathrm{d}\omega\wedge\theta+\overline{\rho}((1,0),(n,\alpha))\omega\wedge\mathrm{d}\theta,$$

and

$$\mathrm{d}^2(\omega\wedge\theta)=\overline{\rho}((1,0),(n+1,\alpha))\mathrm{d}\omega\wedge\mathrm{d}\theta+\overline{\rho}((1,0),(n,\alpha))\mathrm{d}\omega\wedge\mathrm{d}\theta=0. \quad (8)$$

Hence

$$\overline{\rho}((1,0),(n+1,\alpha))+\overline{\rho}((1,0),(n,\alpha))=0. \quad (9)$$

From these relations it follows that

$$\overline{\rho}((1,0),(n,\alpha))=(-1)^n\varphi(\alpha),$$

where $\varphi\colon G\to U(k)$ is the group homomorphism $\varphi(\alpha)=\overline{\rho}((1,0),(0,\alpha))$. From the properties of the cocycle ρ,

$$\overline{\rho}((n,\alpha),(m,\beta))=(-1)^{nm}\varphi^{-m}(\alpha)\varphi^n(\beta)\rho(\alpha,\beta) \quad (10)$$

for any $n,m\in\mathbb{Z}$ and $\alpha,\beta\in G$.

PROPOSITION 1.

Let A be a ρ-algebra with the cocycle ρ. Then any cocycle $\overline{\rho}$ on the group $\overline{G}$ with the conditions $\overline{\rho}|_{G\times G}=\rho$ and (9) are given by (10) for some homomorphism $\varphi\colon G\to U(k)$.

We will denote below $\Omega^*(A,\varphi)$ or simply $\Omega^*(A)$ the $\overline{G}$-graded algebra of forms with the cocycle $\overline{\rho}$ and the derivation $d = d_\varphi$ of degree (1, 0).

Therefore for any ρ-algebra A, a group homomorphism $\varphi\colon G \to U(k)$ and an element $\alpha \in G$, we have the complex:

$$0 \to A_\alpha \xrightarrow{d_\varphi} \Omega^1_\alpha(A,\varphi) \xrightarrow{d_\varphi} \Omega^2_\alpha(A,\varphi) \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} \Omega^i_\alpha(A,\varphi) \xrightarrow{d_\varphi} \Omega^{i+1}_\alpha(A,\varphi) \xrightarrow{d_\varphi} \cdots.$$

The cohomology of this complex term $\Omega^i_\alpha(A,\varphi)$ is denoted by $H^i_\alpha(A,\varphi)$ and will be called as the *de Rham cohomology of the ρ-algebraA.*

PROPOSITION 2.

Let $f\colon A \to B$ be a homomorphism of degree $\alpha \in G$ between the G-graded ρ-algebras. There is a natural homomorphism $\Omega(f)\colon \Omega^*(A) \to \Omega^*(B)$ which in degree n is $\Omega(f)\colon \Omega^n_\beta(A) \to \Omega^n_{\beta+(n+1)\alpha}(A)$ and has the G'-degree $(0,(n+1)\alpha)$ given by

$$\Omega^n(f)(a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n) = f(a_0)\mathrm{d}f(a_1) \wedge \cdots \wedge \mathrm{d}f(a_n). \tag{11}$$

4. Frölicher–Nijenhuis bracket of ρ-algebras

4.1 *Derivations*

Here we present the Frölicher–Nijenhuis calculus over the algebra of forms defined in the previous section.

Denote by $\mathrm{Der}_{(k,\alpha)}(\Omega^*(A))$ the space of derivations of degree (k,α) i.e. an element $D \in \mathrm{Der}_{(k,\alpha)}(\Omega^*(A))$ satisfies the relations:

1) D is linear,
2) the G'-degree of D is $|D| = (k,\alpha)$, and
3) $D(\omega \wedge \theta) = D\omega \wedge \theta + \overline{\rho}((k,\alpha),(n,\beta))\omega \wedge D\theta$ for any $\theta \in \Omega^n_\beta(A)$.

Theorem 2. *The space $\overline{\rho}$-$\mathrm{Der}\ \Omega^*(A) = \oplus_{(k,\alpha)\in\overline{G}} \mathrm{Der}_{(k,\alpha)}\ \Omega^*(A)$ is a $\overline{\rho}$-Lie algebra with the bracket $[D_1,D_2] = D_1 \circ D_2 - \overline{\rho}(|D_1|,|D_2|)D_2 \circ D_1$.*

4.2 *Fields*

Let us denote by $\mathscr{L}\colon \mathrm{Hom}^A_A(\Omega^1(A),A) \to \rho$-Der (A) the isomorphism from Theorem 1. We also denote by $\mathfrak{X}(A) := \mathrm{Hom}^A_A(\Omega^1(A),A)$ the space of fields of the algebra A. Then $\mathscr{L}\colon \mathfrak{X}(A) \to \rho$-Der $(A;A)$ is an isomorphism of vector G-graded spaces. The space of ρ-derivations ρ-Der (A) is a Lie ρ-algebra with the ρ-bracket $[\cdot,\cdot]$, and so we have an induced ρ-Lie bracket on $\mathfrak{X}(A)$ which is given by

$$\mathscr{L}([X,Y]) = [\mathscr{L}_X,\mathscr{L}_Y] = \mathscr{L}_X\mathscr{L}_Y - \rho(|X|,|Y|)\mathscr{L}_Y\mathscr{L}_X \tag{12}$$

and will be referred to as the ρ-Lie bracket of fields.

Lemma 1. *Each field $X \in \mathfrak{X}(A)$ is by definition an A-bimodule homomorphism $\Omega_1(A) \to A$ and it prolongs uniquely to a graded $\overline{\rho}$-derivation $j(X) = j_X\colon \Omega(A) \to \Omega(A)$ of degree $(-1,|X|)$ by*

$$j_X(a) = 0 \quad \textit{for } \; a \in A = \Omega^0(A),$$
$$j_X(\omega) = X(\omega) \quad \textit{for } \; \omega \in \Omega^1(A)$$

and

$$\begin{aligned} & j_X(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) \\ & \quad = \sum_{i=1}^{k-1} \overline{\rho}\left((-1,|X|),\left(i-1,\sum_{j=1}^{i-1}|\omega_i|\right)\right)\omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge X(\omega_i) \\ & \qquad \times \omega_{i+1} \wedge \cdots \wedge \omega_k + \overline{\rho}\left((-1,|X|),\left(k-1,\sum_{j=1}^{k-1}|\omega_i|\right)\right) \\ & \qquad \times \omega_1 \wedge \cdots \wedge \omega_{k-1} X(\omega_k) \end{aligned}$$

for any $\omega_i \in \Omega^1_{|\omega_i|}(A)$. The $\overline{\rho}$-derivation j_X is called the contraction operator of the field X.

Proof. This is an easy computation. □

With some abuse of notation we also write $\omega(X) = X(\omega) = j_X(\omega)$ for $\omega \in \Omega^1(A)$ and $X \in \mathfrak{X}(A) = \mathrm{Hom}^A_A(\Omega^1(A), A)$.

4.2.1 *Algebraic derivations:* A $\overline{\rho}$-derivation $D \in \mathrm{Der}_{(k,\alpha)}\Omega(A)$ is called *algebraic* if $D|_{\Omega^0(A)} = 0$. Then $D(a\omega) = \overline{\rho}((k,\alpha),(0,|a|))aD(\omega)$ and $D(\omega a) = D(\omega)a$ for any $a \in A_{|a|}$ and $\omega \in \Omega(A)$. It results that D is an A-bimodule homomorphism. We denote by $\mathrm{Hom}_\alpha(\Omega_l(A), \Omega_{k+l}(A))$ the space of A-bimodule homomorphisms from $\Omega_{(l,\alpha)}(A)$ to $\Omega_{(l+k,\alpha)}(A)$ of degree (k,α). Then an algebraic derivation D of degree (k,α) is from $\mathrm{Hom}_\alpha(\Omega_l(A), \Omega_{k+l}(A))$. We denote by $\overline{\rho}\text{-}\mathrm{Der}^{\mathrm{alg}}_{(k,\alpha)}\Omega^*(A)$ the space of all $\overline{\rho}$-algebraic derivations of degree (k,α) from $\Omega^*(A)$. Since D is a $\overline{\rho}$-derivation, D has the following expression on the product of 1-forms $\omega_i \in \Omega^1_{|\omega_i|}(A)$:

$$\begin{aligned} D(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) = & \sum_{i=1}^{k} \overline{\rho}\left(|D|,\left(i-1,\sum_{j=1}^{i-1}|\omega_i|\right)\right) \\ & \times \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge D(\omega_i) \wedge \cdots \wedge \omega_k \end{aligned}$$

and the derivation D is uniquely determined by its restriction on $\Omega^1(A)$,

$$K := D|_{\Omega_1(A)} \in \mathrm{Hom}_\alpha(\Omega_1(A), \Omega_{k+1}(A)). \tag{13}$$

We write $D = j(K) = j_K$ to express this dependence. Note that $j_K(\omega) = K(\omega)$ for $\omega \in \Omega_1(A)$. Next we will use the following notations:

$$\Omega^1_{(k,\alpha)} = \Omega^1_{(k,\alpha)}(A) := \mathrm{Hom}_\alpha(\Omega_1(A), \Omega_k(A)),$$
$$\Omega^1_* = \Omega^1_*(A) = \bigoplus_{k \geq 0, \alpha \in G} \Omega^1_{(k,\alpha)}(A).$$

Elements of the space $\Omega^1_{(k\alpha)}$ will be called *field-valued* (k,α)*-forms*.

4.2.2 *Nijenhuis bracket:*

Theorem 3. *The map* $j\colon \Omega^1_{(k+1,\alpha)}(A) \to \overline{\rho}\text{-}\mathrm{Der}^{\mathrm{alg}}_{(k,\alpha)}\Omega^*(A)$, $K \mapsto j_K$ *defined by*

$$\begin{aligned} j_K(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) \\ = \sum_{i=1}^{k} \overline{\rho}\left((k+1,\alpha),\left(i-1,\sum_{j=1}^{i-1}|\omega_i|\right)\right) \\ \times \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge j_K(\omega_i) \wedge \cdots \wedge \omega_k \end{aligned}$$

is an isomorphism and satisfies the following properties:

1) $j_K\colon \Omega_{(n,\beta)}(A) \to \Omega_{(n+k,\alpha+\beta)}(A)$.
2) $j_K(\omega \wedge \theta) = j_K\omega \wedge \theta + \overline{\rho}((k,\alpha),(n,\beta))\omega \wedge j_K\theta$ *for any* $\theta \in \Omega^n_\beta(A)$.
3) $j_K(a) = 0$ *and* $j_K(\omega) = K(\omega)$ *for any* $\omega \in \Omega^1(A)$.

The module of $\overline{\rho}$-algebraic derivations is obviously closed with respect to the $\overline{\rho}$-commutator of derivations.

Therefore we get a $\overline{\rho}$-Lie algebra structure on

$$\mathscr{N}_{ij}(A) = \bigoplus_{(k,\alpha)\in\overline{G}} \overline{\rho}\text{-}\mathrm{Der}^{\mathrm{alg}}_{(k,\alpha)}\Omega^*(A)$$

which is called the *Nijenhuis algebra* of the ρ-algebra A and its bracket is the $\overline{\rho}$*-Nijenhuis bracket*.

By definition, the Nijenhuis bracket of the elements $K \in \mathrm{Hom}_\alpha(\Omega^1(A),\Omega^{1+k}(A))$ and $L \in \mathrm{Hom}_\beta(\Omega^1(A),\Omega^{1+l}(A))$ is given by the formula

$$[K,L]^\Delta = j_K \circ L - \overline{\rho}((k,\alpha),(l,\beta))\, j_L \circ K$$

or

$$[K,L]^\Delta(\omega) = j_K(L(\omega)) - (-1)^{kl}\varphi^{-l}(\alpha)\varphi^k(\alpha)\rho(\alpha,\beta)\, j_L K(\omega) \tag{14}$$

for all $\omega \in \Omega^1(A)$.

4.2.3 *The Frölicher–Nijenhuis bracket:* The exterior derivative d is an element of $\overline{\rho}\text{-}\mathrm{Der}_{(1,0)}\Omega^*(A)$. In the view of the formula $\mathscr{L}_X = [j_X,d]$ for fields X we define $K \in \Omega^1_{(k,\alpha)}(A)$ the Lie derivation $\mathscr{L}_K = \mathscr{L}(K) \in \overline{\rho}\text{-}\mathrm{Der}_{(k,\alpha)}\Omega^*(A)$ by $\mathscr{L}_K := [j_K,d]$. Then the mapping $\mathscr{L}\colon \Omega^1_* \to \overline{\rho}\text{-}\mathrm{Der}\,\Omega(A)$ is injective by the universal property of $\Omega^1(A)$, since $\mathscr{L}_K(a) = j_K(da) = K(da)$ for $a \in A$.

Theorem 4. *For any* $\overline{\rho}$*-derivation* $D \in \overline{\rho}\text{-}\mathrm{Der}_{(k,\alpha)}\Omega^*(A)$*, there are unique homomorphisms* $\Omega^1_{(k,\alpha)}(A)$ *and* $L \in \Omega^1_{(k+1,\alpha)}(A)$ *such that*

$$D = \mathscr{L}_K + j_L. \tag{15}$$

We have $L = 0$ *if and only if* $[D,d] = 0$*.* D *is algebraic if and only if* $K = 0$*.*

Proof. The map $D|_A\colon a \mapsto D(a)$ is a ρ-derivation of degree α so $D|_A\colon A \to \Omega_{(k,\alpha)}(A)$ has the form $K \circ d$ for an unique $K \in \Omega^1_{(k,\alpha)}(A)$. The defining equation for K is $D(a) - j_K da = \mathscr{L}_K(a)$ for $a \in A$. Thus $D - \mathscr{L}_K$ is an algebraic derivation, so $D - \mathscr{L}_K = j_L$ for an unique $L \in \Omega^1_{(k+1,\alpha)}(A)$.

By the Jacobi identity, we have

$$0 = [j_K, [d,d]] = [[j_K, d], d] + \overline{\rho}((k,\alpha),(1,0))[d,[j_K,d]]$$

so $2[\mathscr{L}_K, d] = 0$. It follows that $[D,d] = [j_L, d] = \mathscr{L}_L$ and using the injectivity of $\mathscr{L}$ results that $L = 0$. □

Let $K \in \Omega^1_{(k,\alpha)}(A)$ and $L \in \Omega^1_{(l,\beta)}(A)$. Definition of the $\overline{\rho}$-Lie derivation results in $[[\mathscr{L}_K, \mathscr{L}_L], d] = 0$ and using the previous theorem results that is a unique element which is denoted by $[K,L] \in \Omega^1_{(k+l,\alpha+\beta)}(A)$ such that

$$[\mathscr{L}_K, \mathscr{L}_L] = \mathscr{L}_{[K,L]} \tag{16}$$

and this element will be the denoted by the abstract Frölicher–Nijenhuis of K and L.

Theorem 5. *The space $\Omega^1_*(A) = \oplus_{(k,\alpha)\in\overline{G}}\Omega^1_{(k,\alpha)}(A)$ with the usual grading and the Frölicher–Nijenhuis is a $\overline{G}$-graded Lie algebra. $\mathscr{L}\colon (\Omega^1_*, [\cdot,\cdot]) \to \overline{\rho}\text{-Der }\Omega(A)$ is an injective homomorphism of $\overline{G}$-graded Lie algebras. For fields in $\mathrm{Hom}^A_A(\Omega^1(A), A)$ the Frölicher–Nijenhuis coincides with the bracket defined in (12).*

4.3 *Naturality of the Frölicher–Nijenhuis bracket*

Let $f\colon A \to B$ be an homomorphism of degree 0 between the G-graded ρ-algebras A and B. Two forms $K \in \Omega^1_{(k,\alpha)}(A) = \mathrm{Hom}_\alpha(\Omega_1(A), \Omega_k(A))$ and $K' \in \Omega^1_{(k,\alpha)}(B) = \mathrm{Hom}_\alpha\,(\Omega_1(B), \Omega_k(B))$ are *f-related* or *f-dependent* if we have

$$K' \circ \Omega^1(f) = \Omega_k(f) \circ K\colon \Omega^1_\alpha(A) \to \Omega^k_\alpha(B)$$

where $\Omega_*(f)\colon \Omega(A) \to \Omega(B)$ is the homomorphism from (11) induced by f.

Theorem 6.

(1) *If K and K' are f-related as above then $j_{K'} \circ \Omega(f) = \Omega(f) \circ j_K\colon \Omega(A) \to \Omega(B)$.*
(2) *If $j_K \circ \Omega(f)|_{d(A)} = \Omega(f) \circ j_K|_{d(A)}$, then K and K' are f-related, where $d(A) \subset \Omega^1(A)$ is the space of exact 1-forms.*
(3) *If K_j and K'_j are f-related for $j = 1,2$ then $j_{K_1} \circ K_2$ and $j_{K'_1} \circ K'_2$ are f-related and also $[K_1,K_2]^\Delta, [K'_1,K'_2]^\Delta$ are f-related.*
(4) *If K and K' are f-related then $\mathscr{L}_{K'} \circ \Omega(f) = \Omega(f) \circ \mathscr{L}_K\colon \Omega(A) \to \Omega(B)$.*
(5) *If $\mathscr{L}_{K'} \circ \Omega(f)|_{\Omega_0(A)} = \Omega(f) \circ \mathscr{L}_K|_{\Omega_0(A)}$ then K and K' are f-related.*
(6) *If K_j and K'_j are f-related for $j = 1,2$ then their Frölicher–Nijenhuis brackets $[K_1,K_2]$ and $[K'_1,K'_2]$ are also f-related.*

Acknowledgement

The author wishes to express his thanks to Prof. Gh. Pitiş for many valuable remarks and for a very fruitful and exciting collaboration.

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